Particle methods
Part 3

The Finite Volume Particle Method (FVPM)

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New CFD should do everything classical CFD can

- **mathematically** accurate and robust

  consistency
  stability
  convergence
  conservation

- **practically** accurate and robust

  validation

- walls
- inlets and outlets
- non-uniform resolution
- adaptive resolution

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Problems with basic SPH

- Low order of numerical convergence
- Sensitivity to particle distribution
- Computational cost
- ~~No natural treatment of boundary conditions~~

The lineage of FVPM

Hietel, Steiner, Struckmeier	2000	***A finite-volume partice method for compressible flows*** 2D, 1st-order
Junk	2001	***Do finite volume methods need a mesh?***
Ismagilov	2005	Smooth volume integral method 1D with MUSCL
Keck, Hietel	2005	Incompressible flow
Nestor *et al.*	2008, 2009	2D with MUSCL, higher order (?), viscous flow, moving body
Quinlan et al.	2011, 2014	Fast exact evaluation of particle volume and area 2D
Jahanbakhsh et al.	2016, 2017	Fast exact evaluation of particle volume and area in 3D

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The finite volume particle method

Conservation law:

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \nabla \bullet \mathbf{F}(\mathbf{U}) = 0$$

Introduce a compactly supported test function $\psi_i(\mathbf{x})$:

weak form:

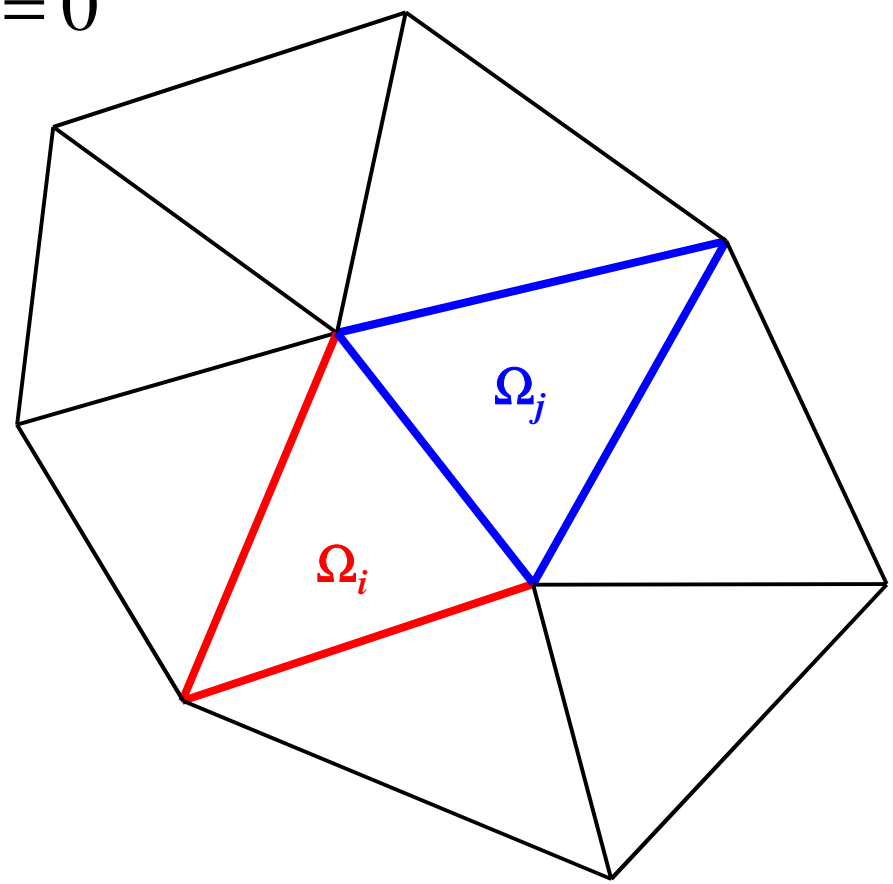
$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \mathrm{d}\mathbf{x} + \int_{\Omega} \psi_i \nabla \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

$$\int_{\Omega} \psi_i \frac{d\mathbf{U}}{dt} d\mathbf{x} - \int_{\Omega} \nabla \psi_i \bullet \mathbf{F}(\mathbf{U}) d\mathbf{x} = 0$$

Choice of test function and support volume

$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \mathrm{d}\mathbf{x} - \int_{\Omega} \nabla \psi_i \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

$$\psi_i(\mathbf{x}) = \begin{cases} 1 & x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}$$

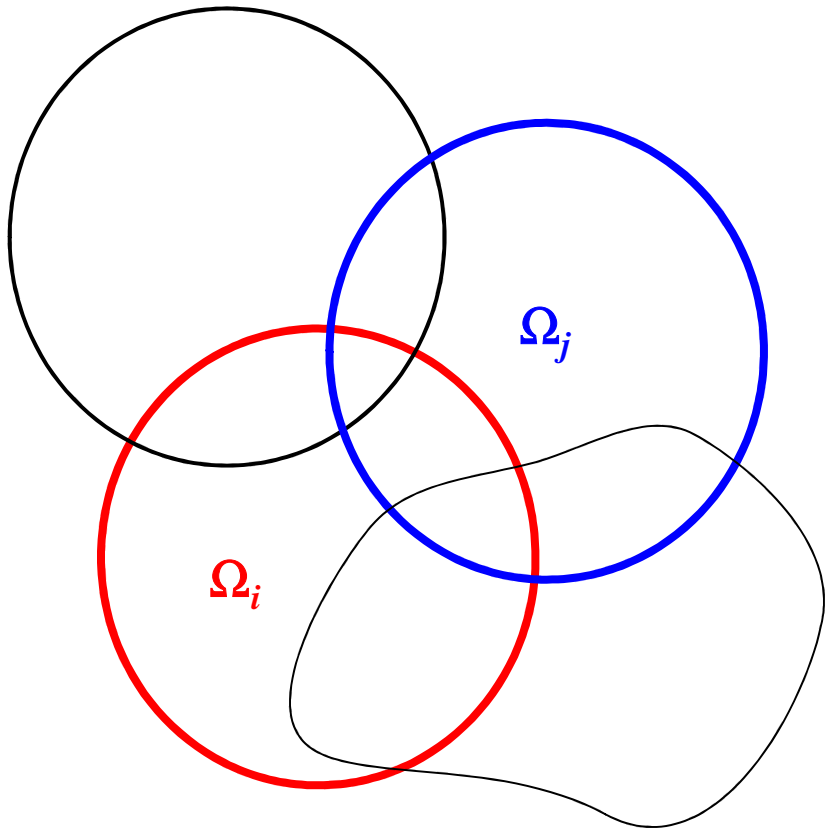


$\rightarrow$ **finite volume method**

Choice of test function and support volume

$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \mathrm{d}\mathbf{x} - \int_{\Omega} \nabla \psi_i \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

$$\psi_i(\mathbf{x}) = \begin{cases} \dfrac{W_i(\mathbf{x})}{\sum_k W_k(\mathbf{x})} & x \in \Omega_i \\ 0 & \text{otherwise} \end{cases}$$



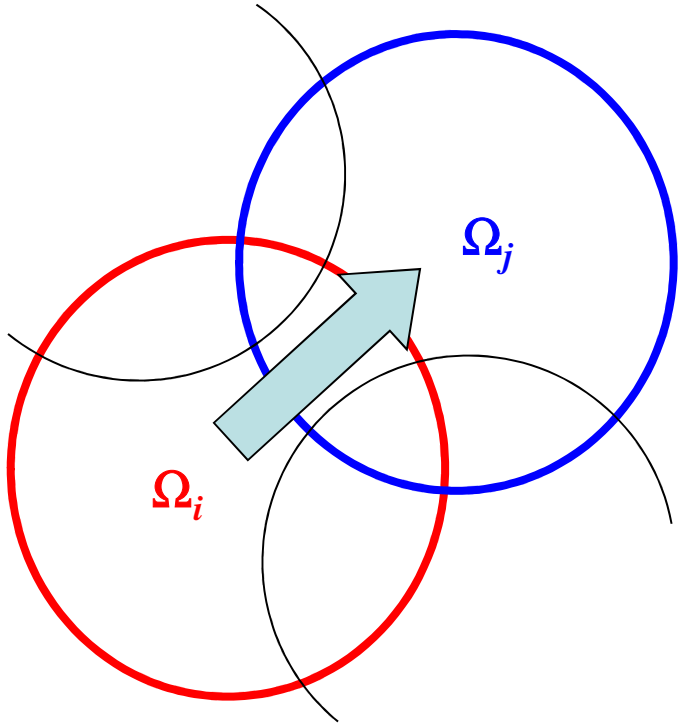
FVPM

→ **finite volume *particle* method**

Interpretation in terms of pair interactions

$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \mathrm{d}\mathbf{x} - \int_{\Omega} \nabla \psi_i \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

$$\sum_j \frac{W_i(\mathbf{x}) \nabla W_j(\mathbf{x}) - W_j(\mathbf{x}) \nabla W_i(\mathbf{x})}{\left(\sum_k W_k(\mathbf{x}) \right)^2}$$



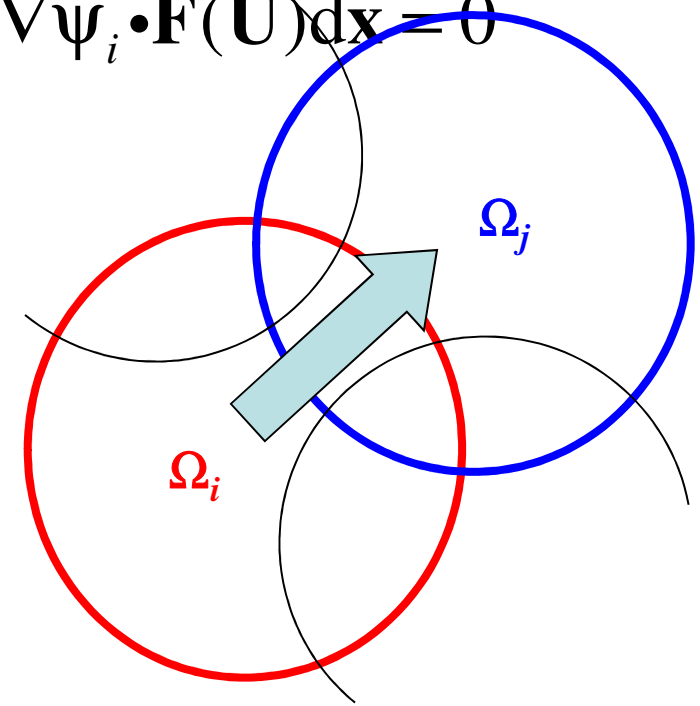
$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \mathrm{d}\mathbf{x} - \sum_j \int_{\Omega} \left(\gamma_{ij} - \gamma_{ji} \right) \bullet \mathbf{F}(\mathbf{U}) \mathrm{d}\mathbf{x} = 0$$

Particle interactions

$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\mathrm{d}\mathbf{x} - \int_{\Omega} \nabla\psi_i \bullet \mathbf{F}(\mathbf{U})\mathrm{d}\mathbf{x} = 0$$

$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\mathrm{d}\mathbf{x} - \int_{\Omega} \nabla\psi_i \bullet \mathbf{F}(\mathbf{U})\mathrm{d}\mathbf{x} = 0$$

$$\sum_j \frac{W_i(\mathbf{x})\nabla W_j(\mathbf{x}) - W_j(\mathbf{x})\nabla W_i(\mathbf{x})}{\left(\sum_k W_k(\mathbf{x})\right)^2}$$



$$\int_{\Omega} \psi_i \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\mathrm{d}\mathbf{x} - \sum_j \int_{\Omega} \left(\gamma_{ij} - \gamma_{ji}\right) \bullet \mathbf{F}(\mathbf{U})\mathrm{d}\mathbf{x} = 0$$

3 numerical approximations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\psi_i \mathbf{U}\mathrm{d}\mathbf{x} - \sum_j \int_{\Omega}\left(\gamma_{ij}-\gamma_{ji}\right)\bullet\mathbf{F}(\mathbf{U})\mathrm{d}\mathbf{x} - \int_{\Omega}\frac{\mathrm{d}\psi_i}{\mathrm{d}t}\mathbf{U}\mathrm{d}\mathbf{x} = 0$$

1 replace the weighted volume average $\mathbf{U}$ with a "particle" value

Represent $\mathbf{F}(\mathbf{U}(\mathbf{x}))$ with a single value for the overlap region

2

$$\frac{\mathrm{d}}{\mathrm{d}t}(V_i\mathbf{U}_i) - \sum_j \beta_{ij}\mathbf{F}_{ij} - \int_{\Omega}\frac{\mathrm{d}\psi_i}{\mathrm{d}t}\mathbf{U}\mathrm{d}\mathbf{x} = 0$$

where

$$V_i = \int_{\Omega}\psi_i \mathrm{d}\mathbf{x}$$

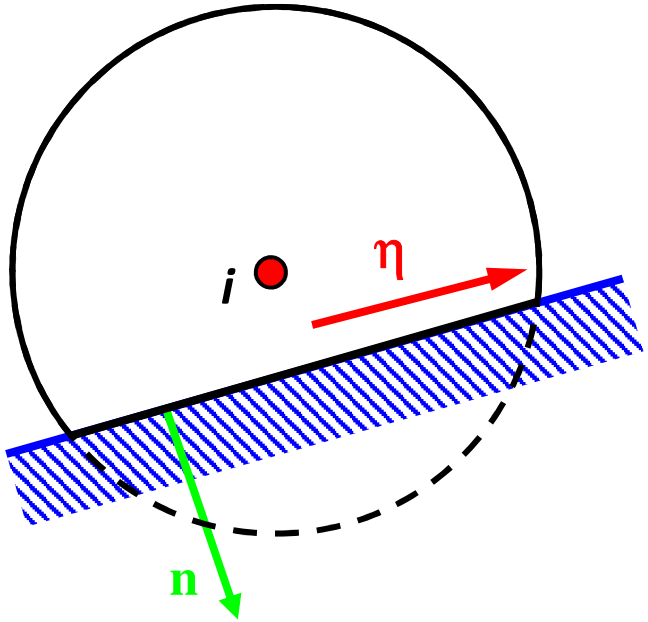
3

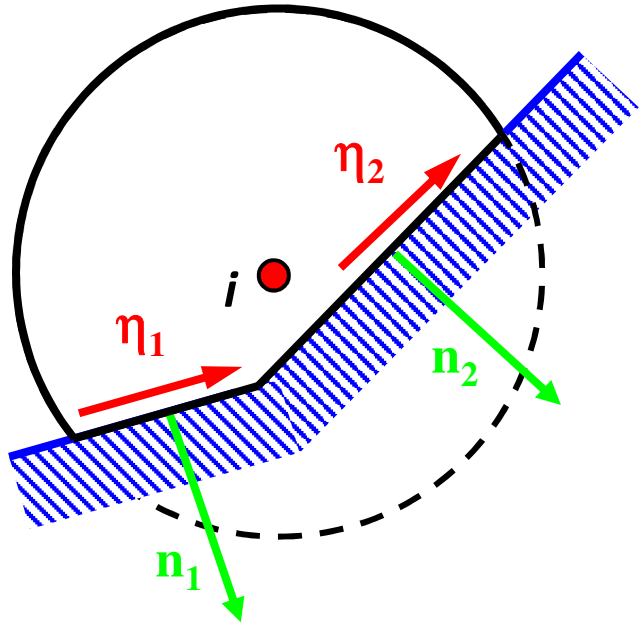
Reconstruct $\mathbf{U}_i$, $\mathbf{U}_j$ at interface for Riemann problem

$$\rightarrow \mathbf{F}_{ij} = \mathbf{F}\left(\mathbf{U}_i, \mathbf{U}_j\right)$$

Boundary conditions

Particle support is truncated at boundary.





Compute boundary interaction vector directly…

$$\beta_i^b = \int \frac{W_i}{\sum_k W_k(\mathbf{x})} \mathbf{n}\, d\boldsymbol{\eta}$$

…or by enforcing

$$\sum_j \beta_{ij} + \beta_i^b = 0$$

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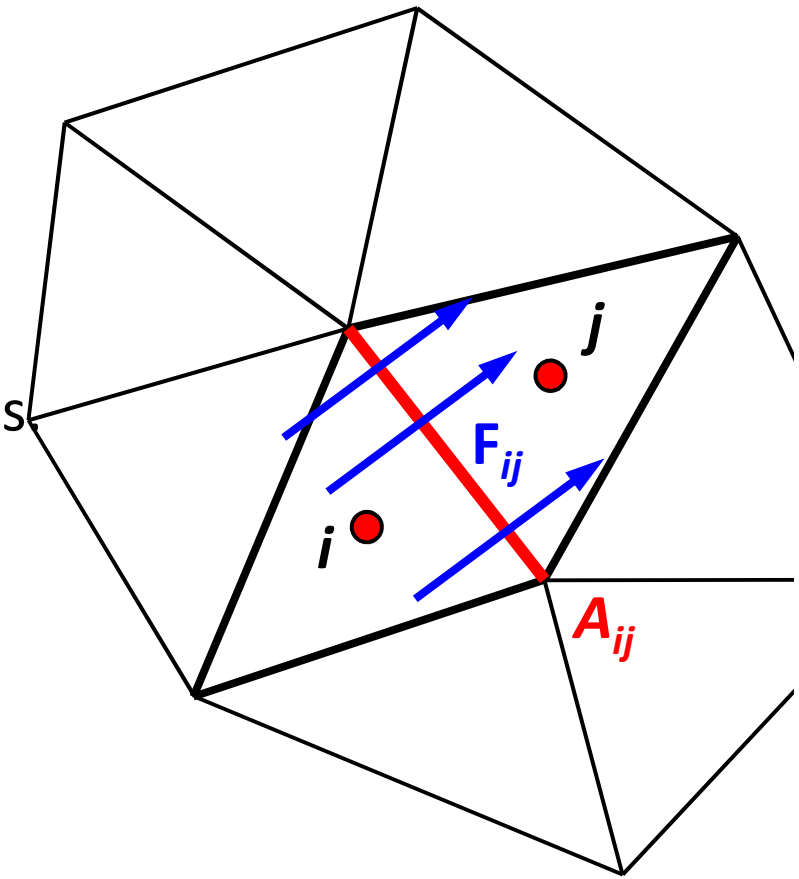
FVM

Classical mesh-based finite volume method: discrete cells

$$\frac{d(V\phi)_i}{dt} \cong \sum_j F_{ij} A_{ij}$$

Exact conservation in cell-cell exchanges



FVPM vs FVM

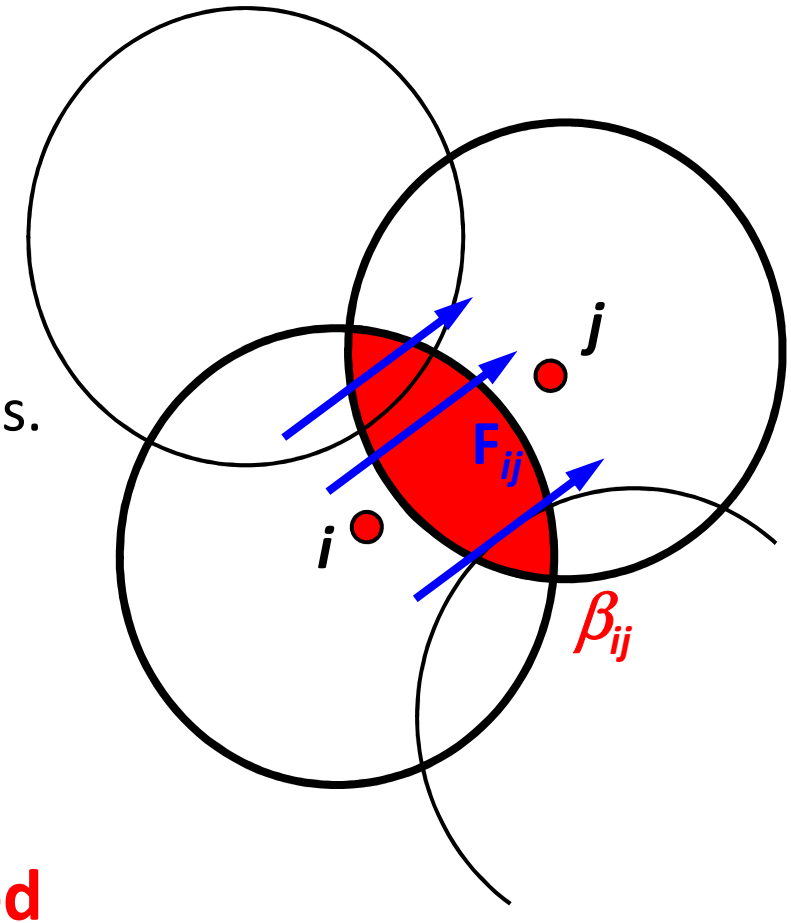
Classical mesh-based finite volume method: discrete cells

$$\frac{d(V\phi)_i}{dt} \cong \sum_j F_{ij} A_{ij}$$

Exact conservation in cell-cell exchanges.

$$\frac{d(V\phi)_i}{dt} \cong \sum_j F_{ij} \beta_{ij}$$

Finite volume particle method: overlapping particles.



The classical finite volume method is a special case of FVPM.

Hietel *et al*. (2000), Junk (2001), Ismagilov (2004), Nestor *et al*. (2008)

FVPM vs FVM

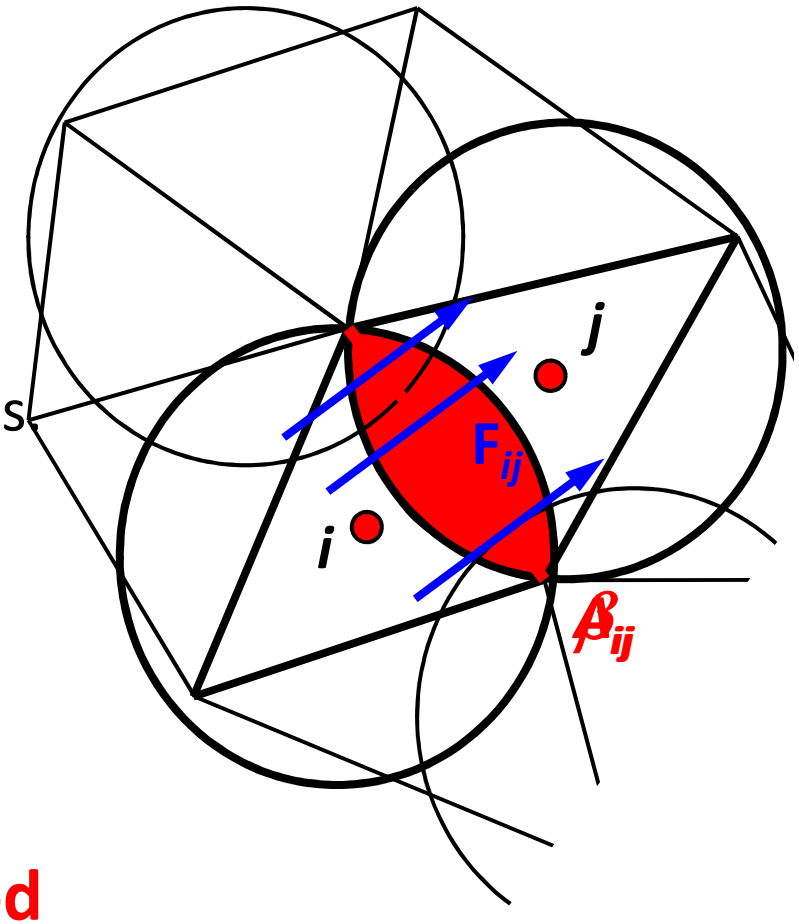
Classical mesh-based finite volume method: discrete cells

$$\frac{d(V\phi)_i}{dt} \cong \sum_j F_{ij} A_{ij}$$

Exact con cell-cell exchanges

Animated versio

$$\frac{d(V\phi)_i}{dt} \cong \sum_j F_{ij} \beta_{ij}$$



Finite volume particle method: overlapping particles.

The classical finite volume method is a special case of FVPM.

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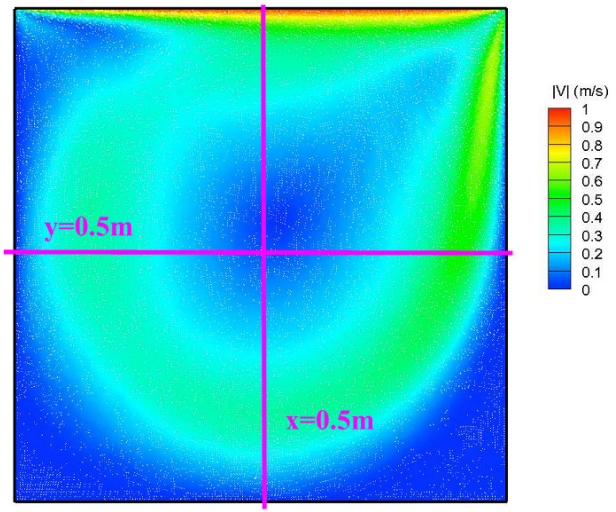
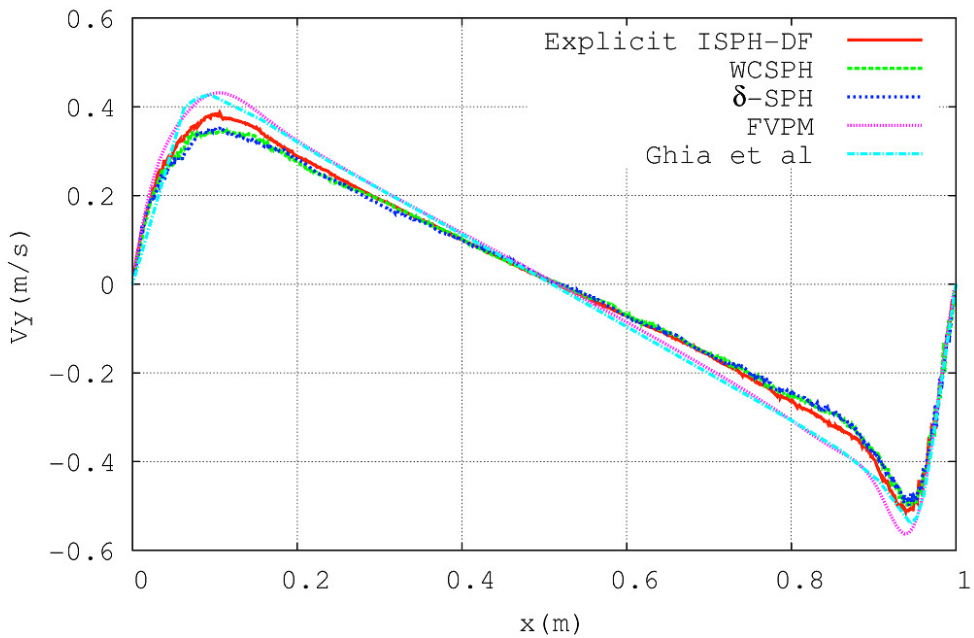
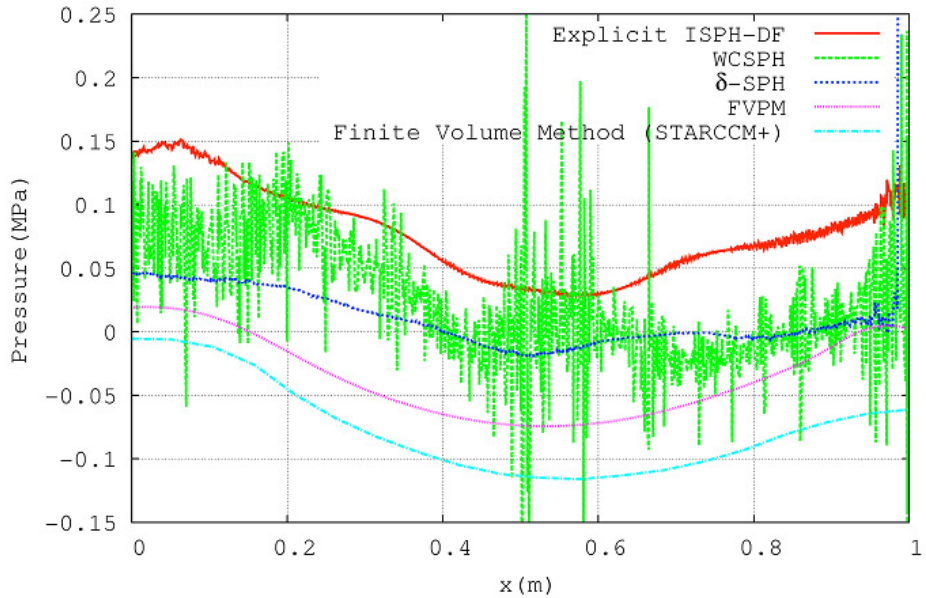
Flux functions, exactness, order of consistency

- Requirements for choice of a flux function – $G(U, U) = F(\mathbf{U})$ and so on

FVPM vs SPH variants







Le Touzé and NextMuSE consortium, SPHERIC Workshop (2012)

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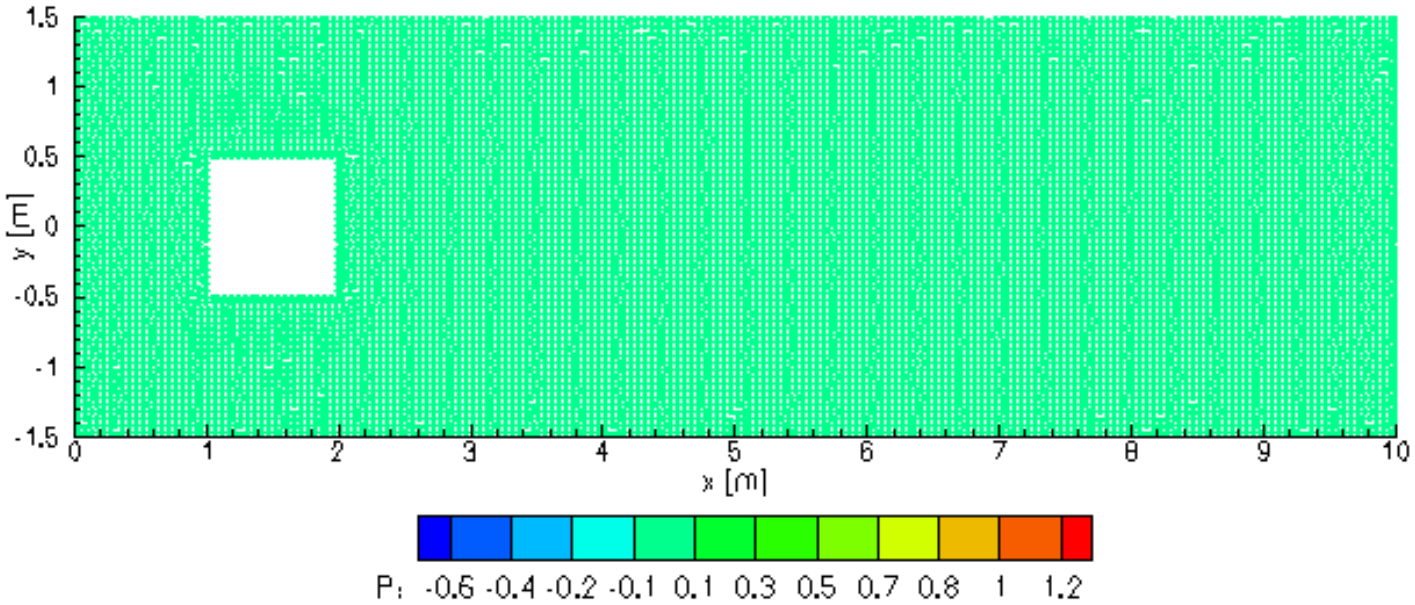
Flow around a body with prescribed motion



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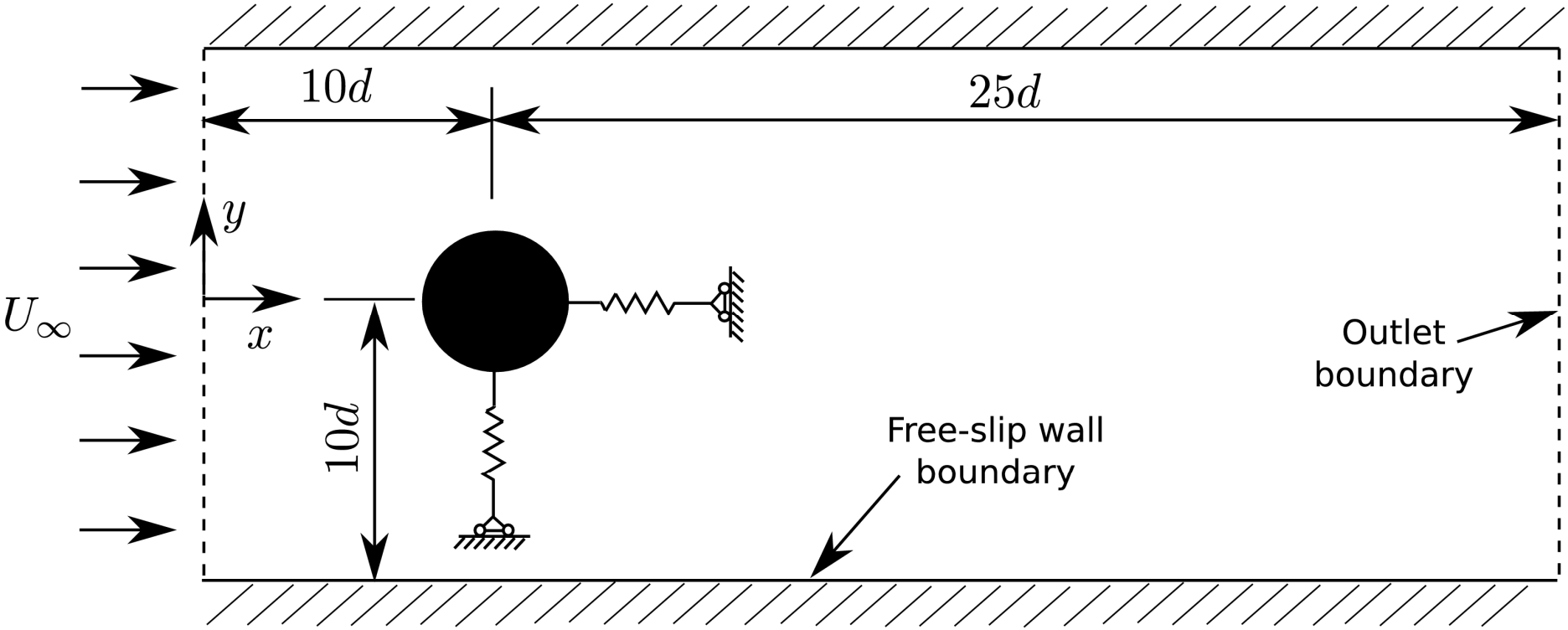
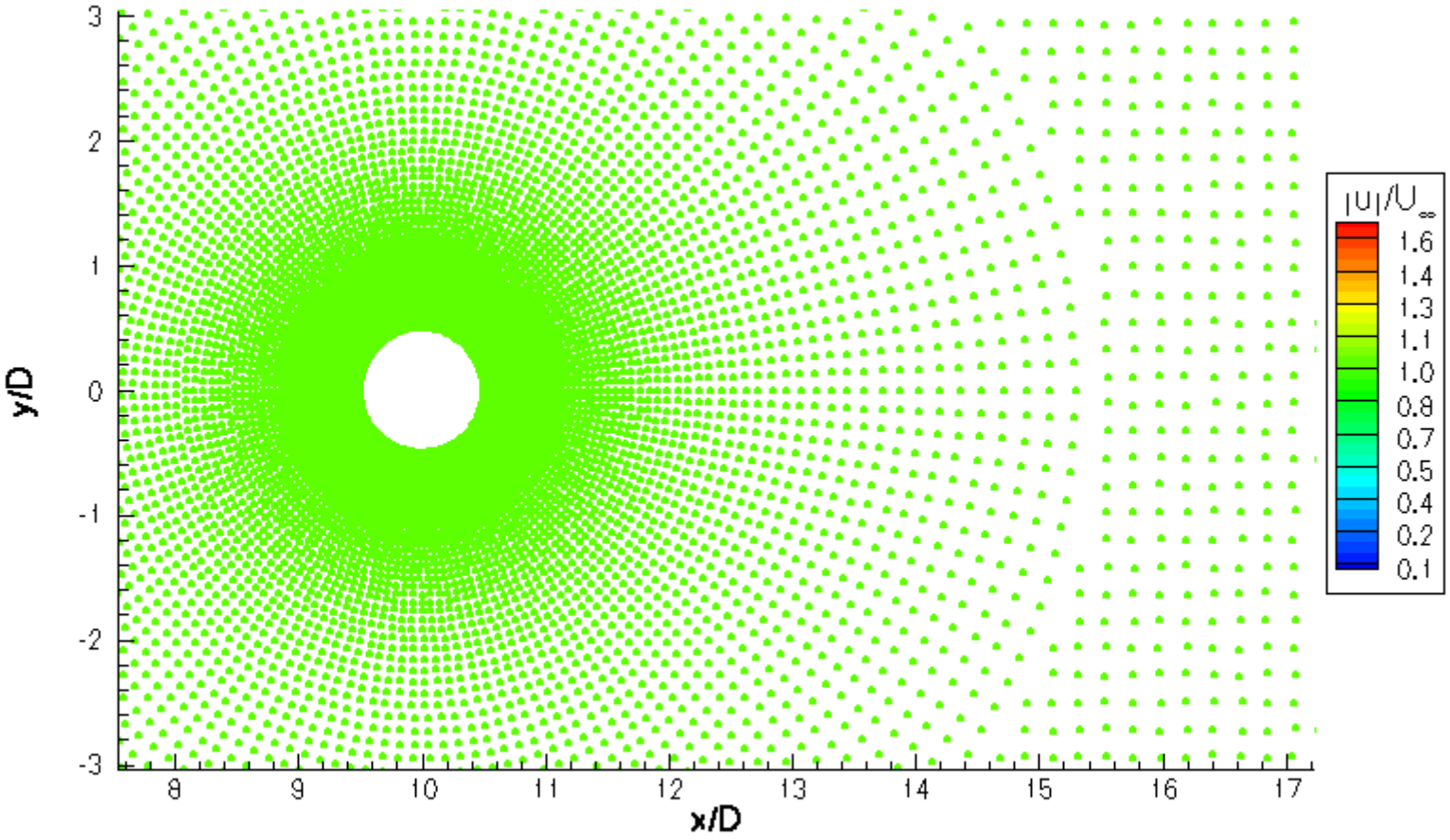
Test case: vortex-induced vibration



Test case: vortex-induced vibration



Nestor (2010)

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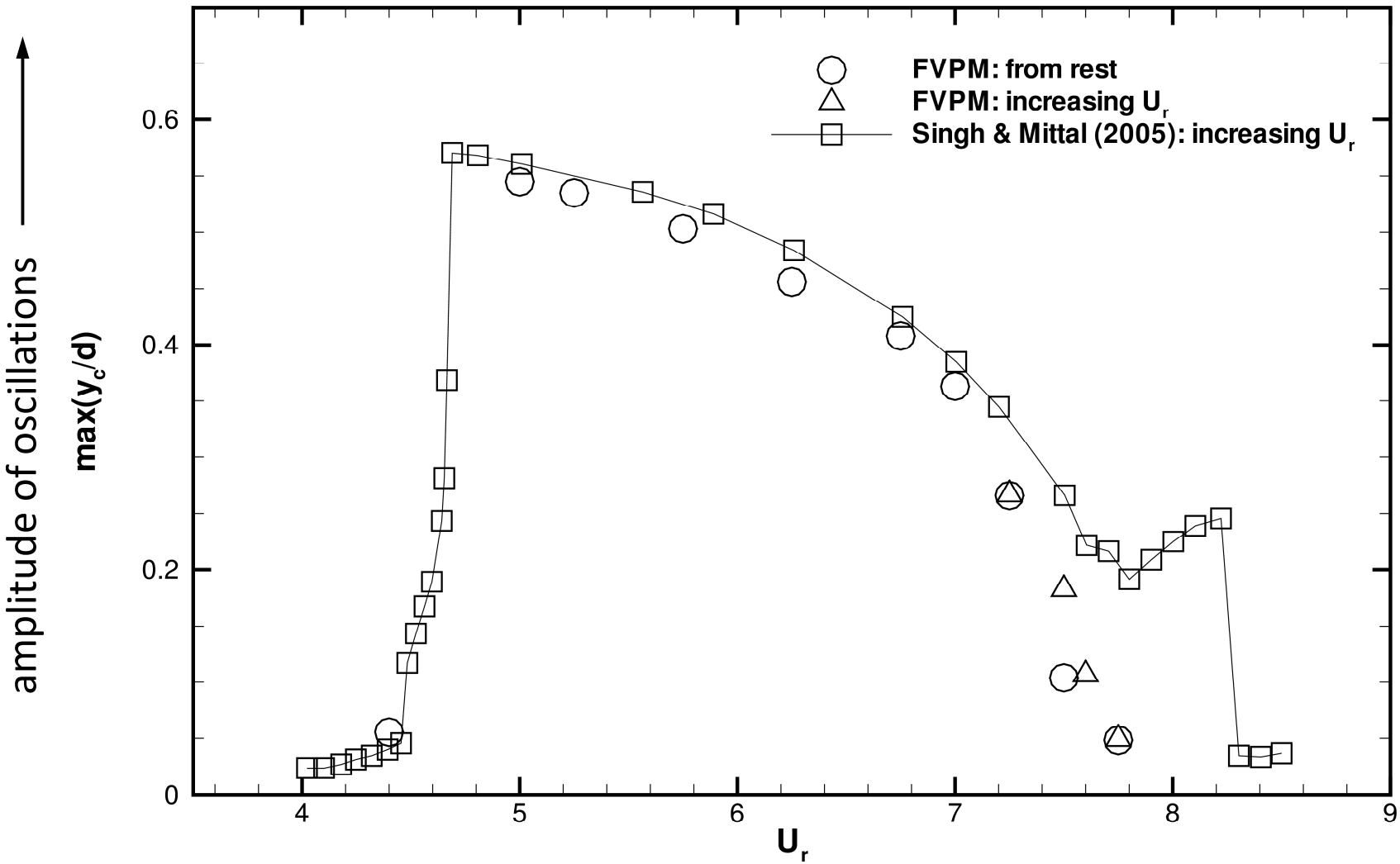
Test case: vortex-induced vibration



$$\frac{\text{vortex-shedding frequency}}{\text{spring-mass natural frequency}}$$

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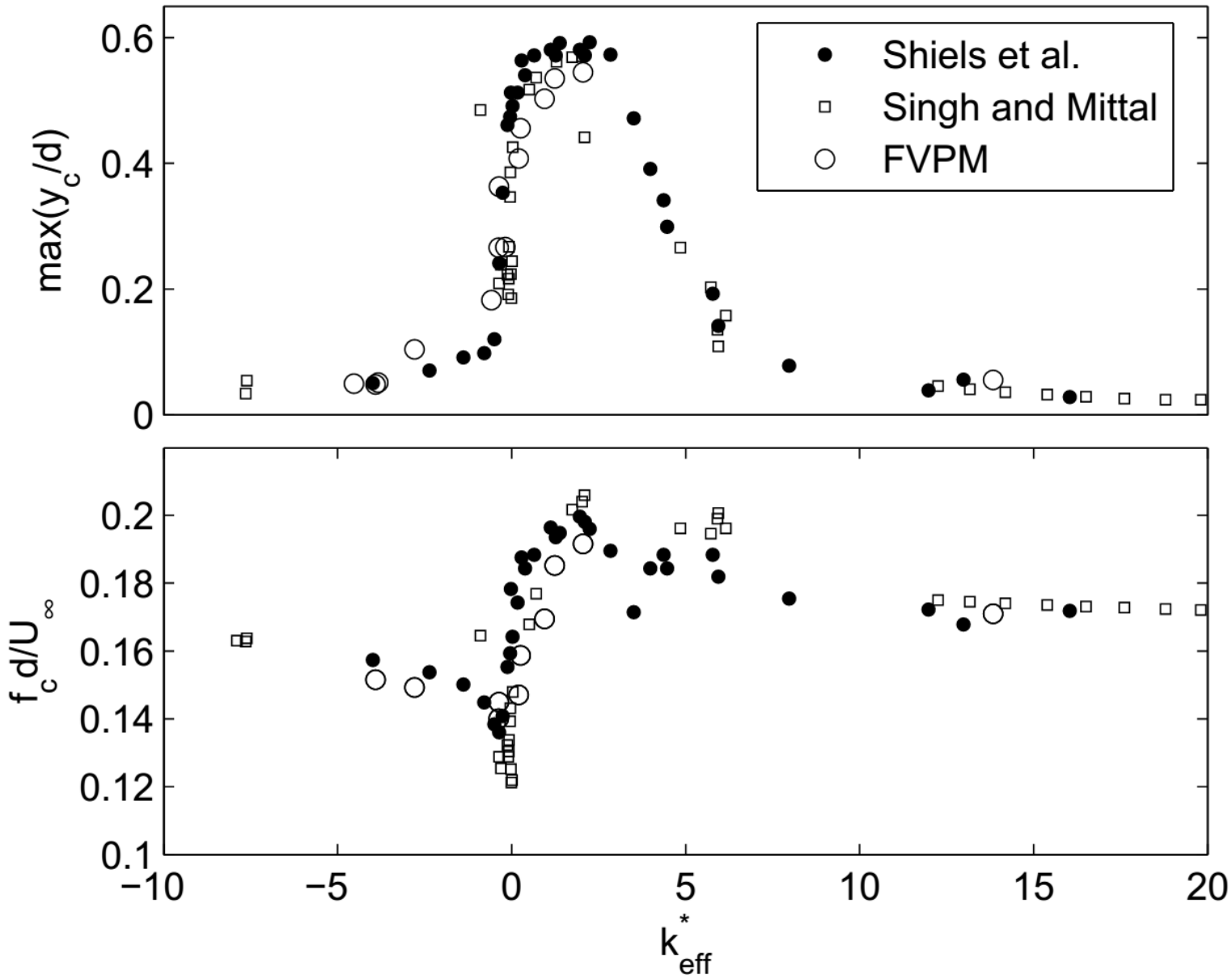
Test case: vortex-induced vibration



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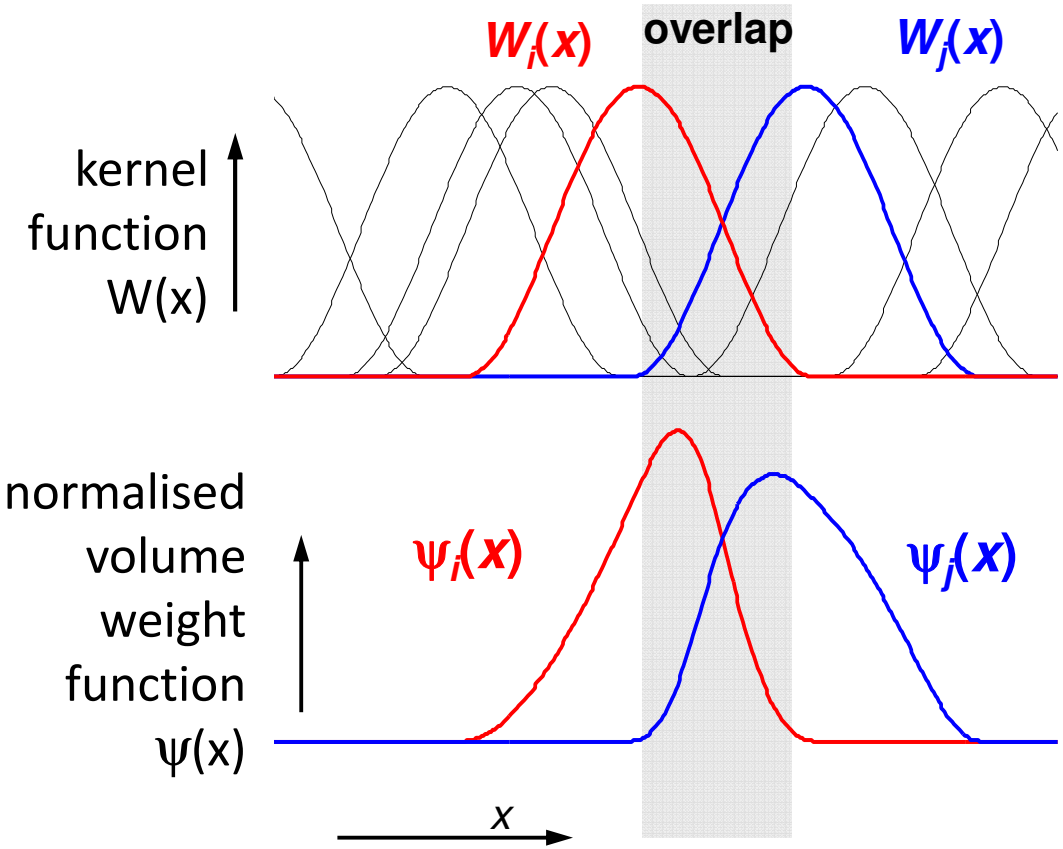
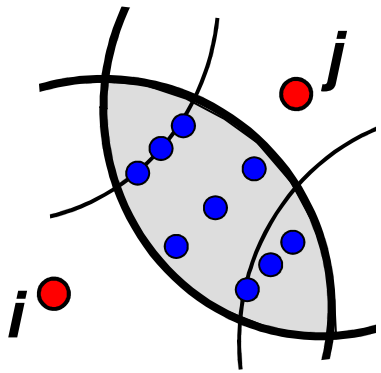
Smooth kernel functions



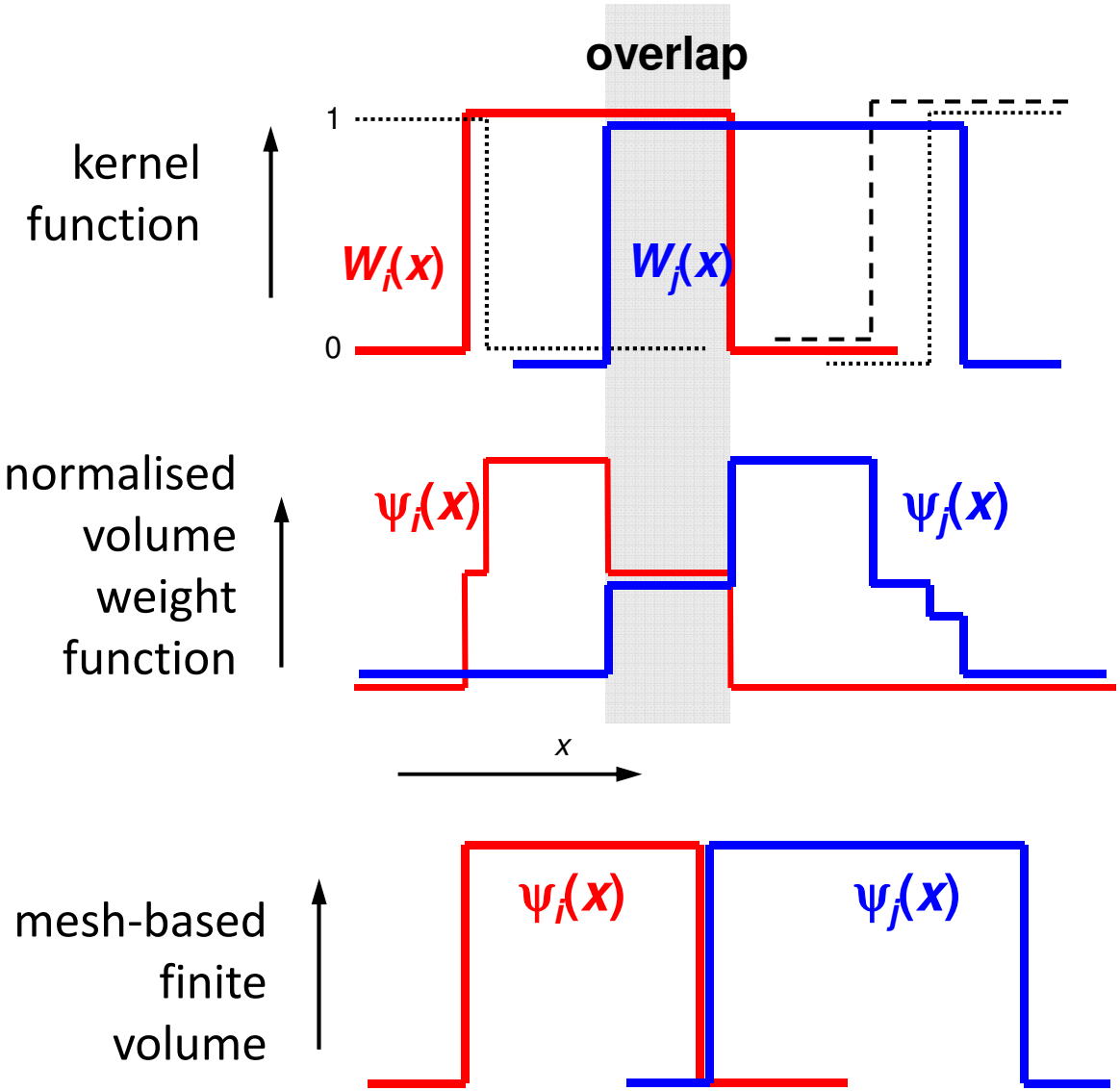
$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left(\sum_k W_k(\mathbf{x})\right)^2} \, \mathrm{d}\,\mathbf{x}$$

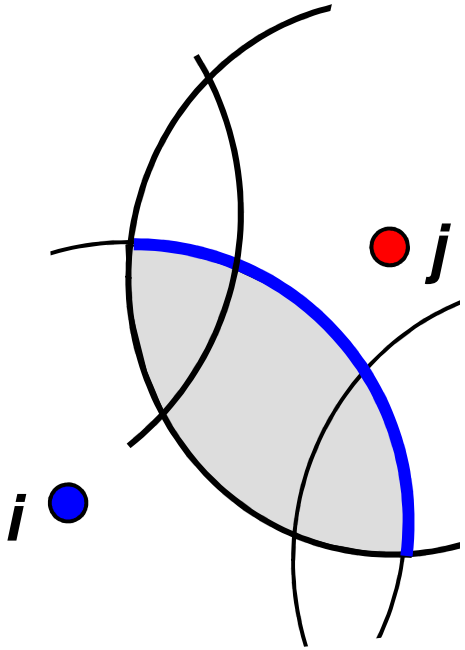


We evaluate β by numerical integration (Gaussian quadrature), and it's **slow.**

Top-hat kernel functions

$$\beta_{ij} = \int \frac{W_i \nabla W_j - W_j \nabla W_i}{\left(\sum_k W_k(\mathbf{x})\right)^2} \mathrm{d}\,\mathbf{x}$$





We only have to integrate on edges (**fast**), and we can do it **exactly**!

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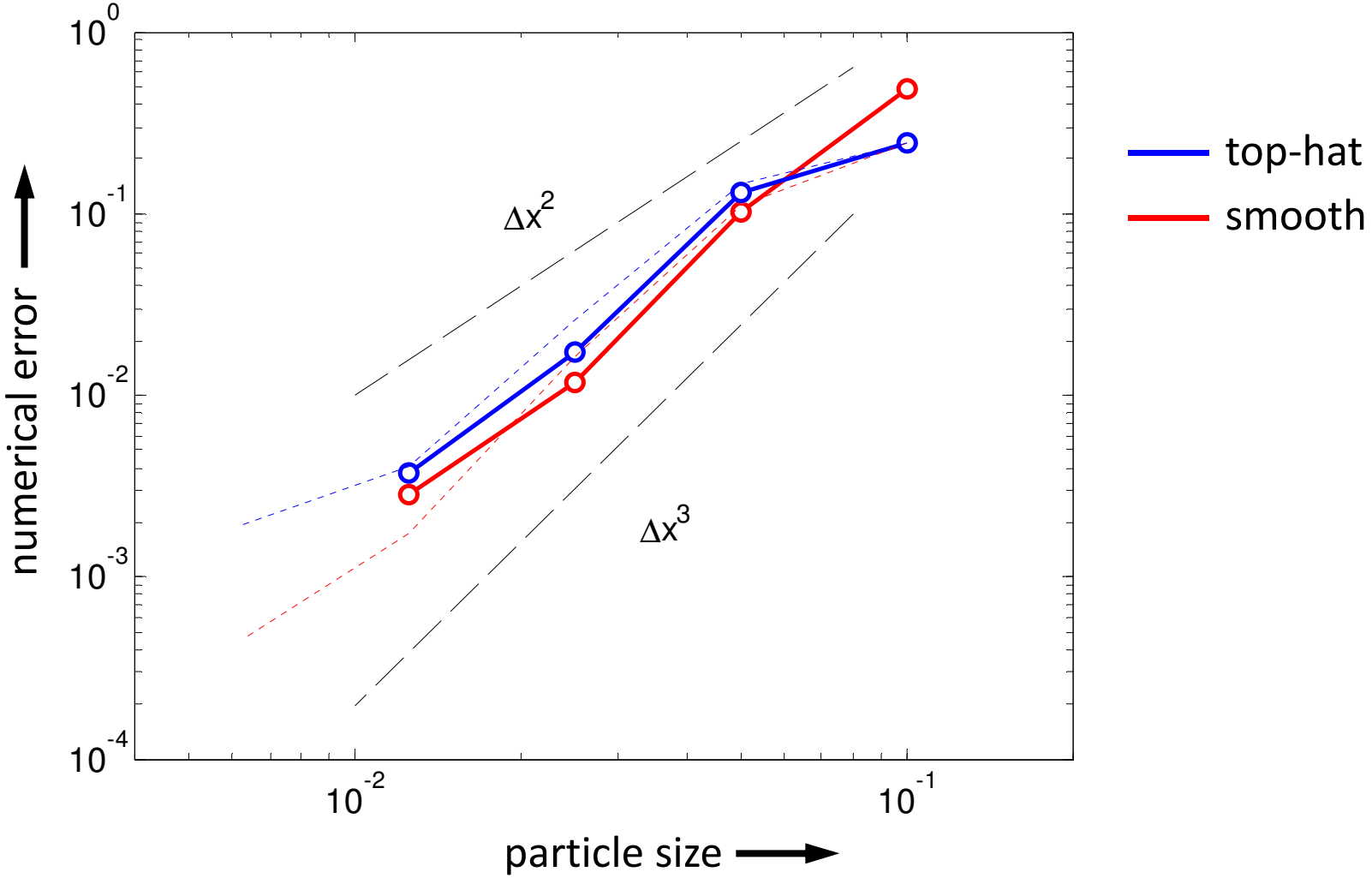
Kernels for FVPM: summary

Top-hat kernels work



speedup ratio 3× to 8×

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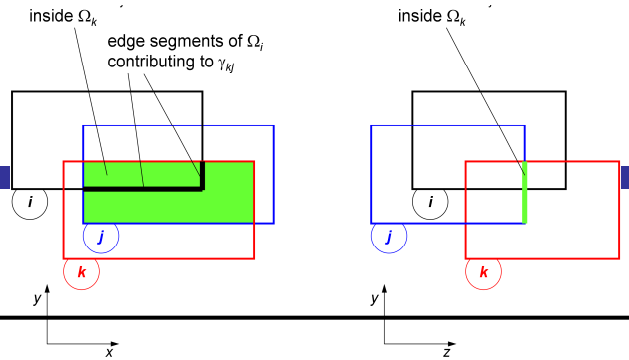
Onwards to 3D!!



2D/3D	integration method	h/Δ*x*	number of neighbours	number of particles	total time	time per particle	time per particle per neighbour
2D	fast	0.26	9	1156	1	1	1
2D	fast	0.51	25	1296	2.7	2.4	0.87
3D	fast	0.26	27	1728	62	41	14
3D	fast	0.51	125	2744	2050	864	173
3D	numerical	0.46	27	2744	187	79	26
3D	numerical	0.76	123	4096	3433	969	197

…or maybe not. At first attempt (c. 2011), 3D “fast”, on cubic particles, is 2-3 orders of magnitude slower than 2D, and not much faster than quadrature.

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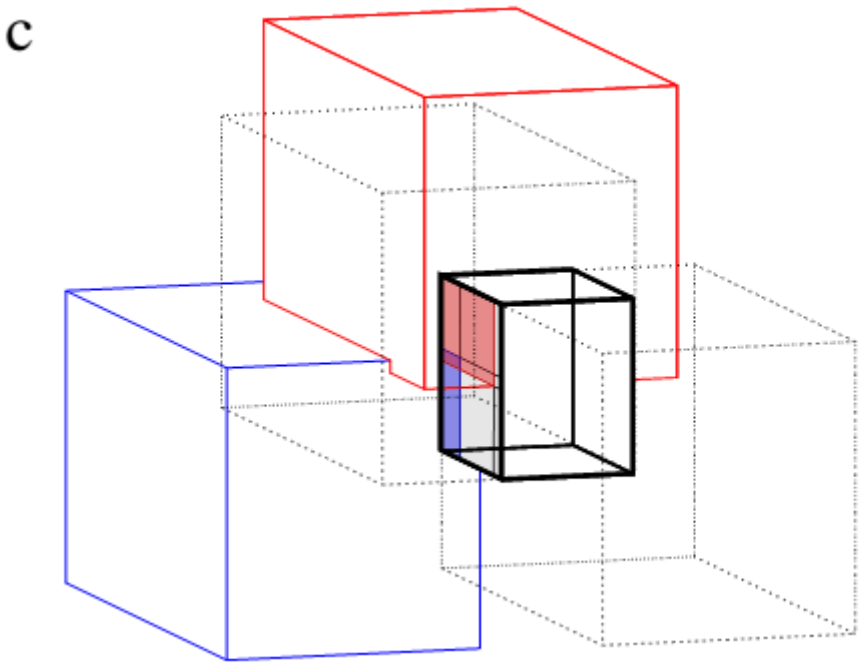
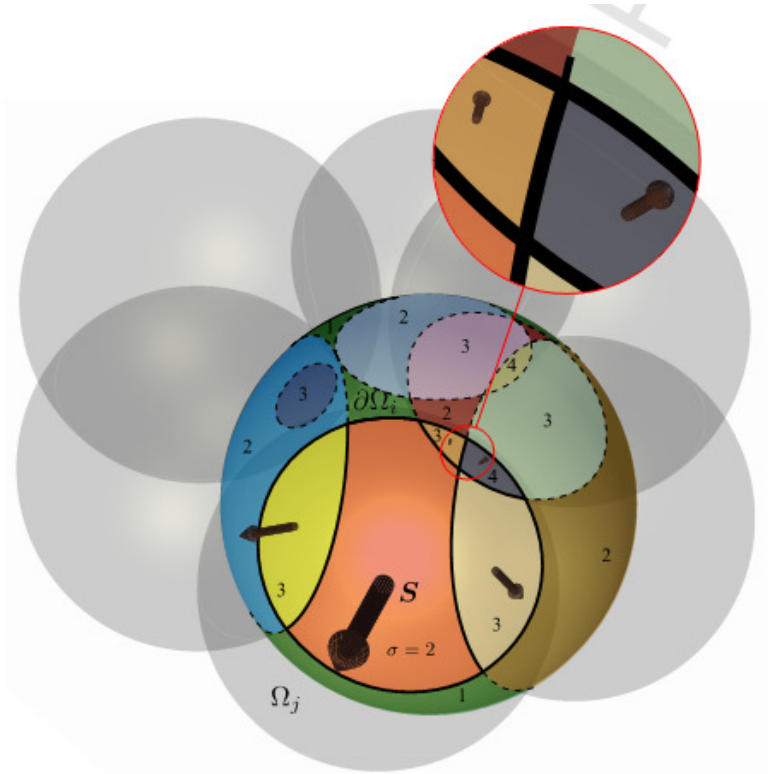
EPFL to the rescue!





Novel algorithms from Jahanbakhsh, Avellan et al. - exact area and volume computations extended to 3D for cubes and spheres, in practically reasonable CPU time

Jahanbakhsh et al. (2016, 2017)

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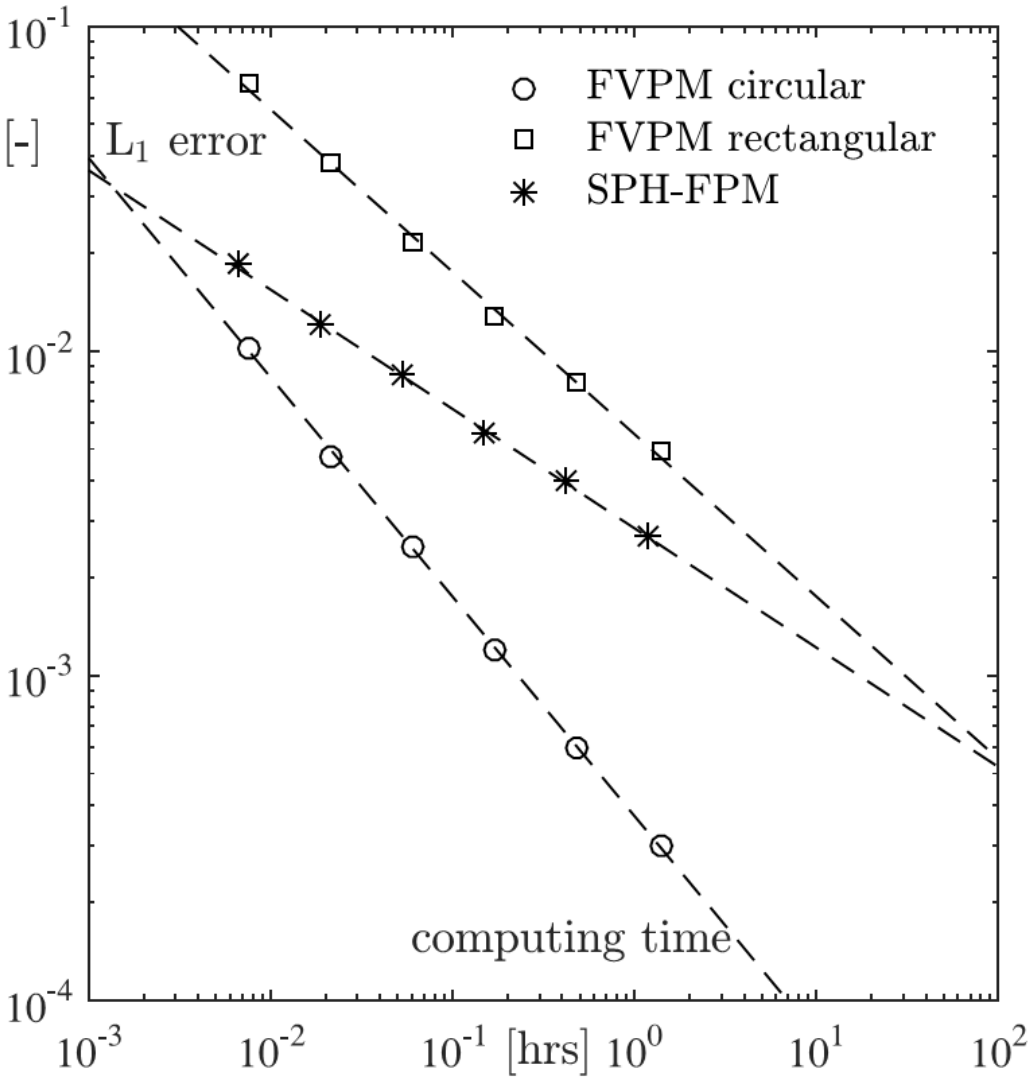
EPFL to the rescue!



Jahanbakhsh et al. (2016, 2017)

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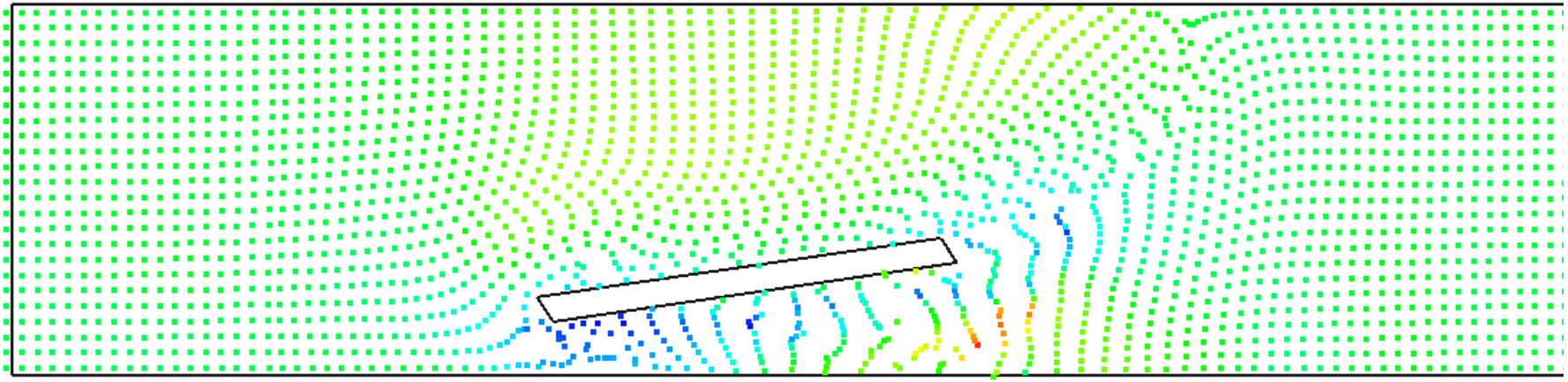
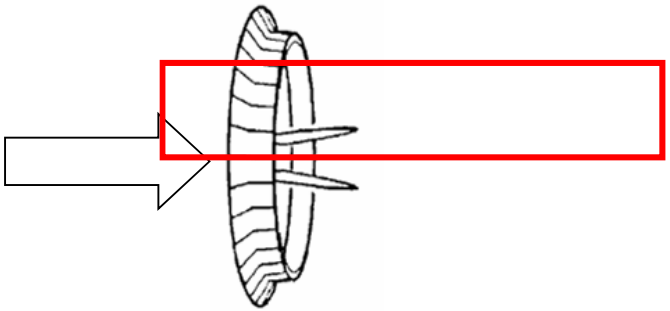
Mechanical heart valve opening





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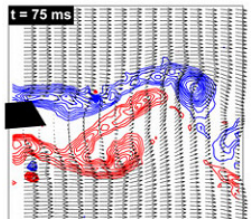
Mechanical heart valve: vortex shedding



Experiment:
Bellofiore *et al.*, *Expts in Fluids* (2011)

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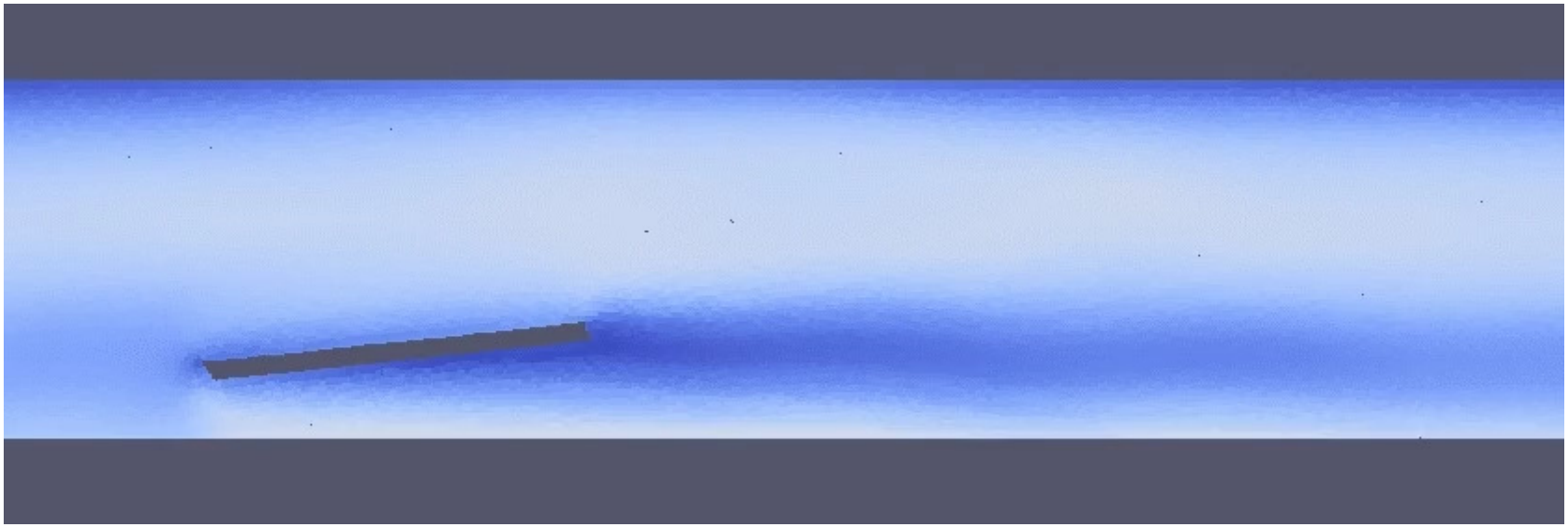
Mechanical heart valve: closing



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